MULTIPLICITY ONE THEOREMS: THE ARCHIMEDEAN CASE

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ABSTRACT. Let G be one of the classical Lie groups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{U}(p,q)$, and let G' be respectively the subgroup $\mathrm{GL}_{n-1}(\mathbb{R})$, $\mathrm{GL}_{n-1}(\mathbb{C})$, $\mathrm{O}(p,q-1)$, $\mathrm{O}_{n-1}(\mathbb{C})$, $\mathrm{U}(p,q-1)$, embedded in G in the standard way. We show that every irreducible Harish-Chandra smooth representation of G' occurs with multiplicity at most one in every irreducible Harish-Chandra smooth representation of G.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Let G be one of the classical Lie groups

$$\mathrm{GL}_n(\mathbb{R}),\ \mathrm{GL}_n(\mathbb{C}),\ \mathrm{O}(p,q),\ \mathrm{O}_n(\mathbb{C}),\ \mathrm{U}(p,q),$$

and let G' be respectively the subgroup

$$\mathrm{GL}_{n-1}(\mathbb{R}),\ \mathrm{GL}_{n-1}(\mathbb{C}),\ \mathrm{O}(p,q-1),\ \mathrm{O}_{n-1}(\mathbb{C}),\ \mathrm{U}(p,q-1),$$

of G, where $p \geq 0$, $q, n \geq 1$. The subgroup G' embeds in G in the usual way as follows. For general linear groups,

$$\mathrm{GL}_{n-1}(\mathbb{K}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_n(\mathbb{K}) \mid a \in \mathrm{GL}_{n-1}(\mathbb{K}) \right\} \subset \mathrm{GL}_n(\mathbb{K}),$$

where $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. The real orthogonal groups are realized as

$$\mathrm{O}(p,q) = \left\{ x \in \mathrm{GL}_{p+q}(\mathbb{R}) \mid x^t\, I_{p,q}\, x = I_{p,q} \right\},$$

where $I_{p,q}$ is the diagonal matrix of size $p + q$ whose first p diagonal entries are 1, and last q diagonal entries are -1. Then

$$\mathrm{O}(p, q-1) = \mathrm{GL}_{p+q-1}(\mathbb{R}) \cap \mathrm{O}(p,q) \subset \mathrm{O}(p,q).$$

Likewise for the complex orthogonal groups and the unitary groups.

By a representation of G, we mean a continuous linear action of G on a complete, locally convex, Hausdorff, complex topological vector space. We say that a representation V of G is a Harish-Chandra smooth representation if it is Fréchet, smooth, of moderate growth, admissible and $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$-finite. Here and as usual, $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ is the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G. The reader may consult [C89] and [W92, Chapter 11] for more details about Harish-Chandra smooth representations.

The main purpose of this paper is to prove the following

Theorem A. *Let V and V' be irreducible Harish-Chandra smooth representations of G and G', respectively. Then the space of G'-intertwining continuous linear maps from V to V' is at most one dimensional, i.e.,*

$$\dim \mathrm{Hom}_{G'}(V, V') \leq 1.$$

Theorem A and its p-adic analog have been expected (first by Bernstein) since the 1980's. When V' is the trivial representation, Theorem A is proved in [AGS1], [AGS2] and [Di], in the case of general linear, orthogonal, and unitary groups, respectively. The p-adic analog of Theorem A is proved in [AGRS] in its full generality.

Remark. *Denote by K the maximal compact subgroup*

$$\mathrm{O}(n),\ \mathrm{U}(n),\ \mathrm{O}(p) \times \mathrm{O}(q),\ \mathrm{O}(n),\ \mathrm{U}(p) \times \mathrm{U}(q)$$

of G, according to the five cases under consideration. Set $K' = G' \cap K$, and denote by $\mathfrak{g}'_{\mathbb{C}}$ the complexified Lie algebra of G'. Given Harish-Chandra smooth representations V and V' of G and G' (respectively), we expect that a certain form of automatic continuity theorem will imply that

$$\mathrm{Hom}_{(\mathfrak{g}'_{\mathbb{C}}, K')}(V_K, V'_{K'}) = \mathrm{Hom}_{G'}(V, V'),$$

where V_K is the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of V, similarly for $V'_{K'}$. Therefore, a $(\mathfrak{g}_{\mathbb{C}}, K)$-module version of Theorem A should hold. Consequently, we expect the theorem to remain true whenever V and V' are irreducible admissible representations.

For any (smooth) manifold M, denote by $C^{-\infty}(M)$ the space of generalized functions on M, which by definition consists of continuous linear functionals on $D_c^{\infty}(M)$, the space of (complex) smooth densities on M with compact supports. The latter is equipped with the usual inductive smooth topology.

By (a version of) the Gelfand-Kazhdan criterion, Theorem A is a consequence of the following result. See Proposition 7.1.

Theorem B. *Let $f \in C^{-\infty}(G)$ satisfy*

$$f(gxg^{-1}) = f(x)$$

for all $g \in G'$. Then we have

$$f(x^{\sigma}) = f(x),$$

where σ is the anti-involution of G given by

$$x^{\sigma} = \begin{cases} x^t, & \text{if } G \text{ is a general linear group,} \\ x^{-1}, & \text{if } G \text{ is an orthogonal group,} \\ \bar{x}^{-1}, & \text{if } G \text{ is a unitary group.} \end{cases}$$

We record another consequence of Theorem B, in the case of general linear groups. As before, let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Denote by $P_n(\mathbb{K})$ the subgroup of $GL_n(\mathbb{K})$ consisting of matrices whose last row is $[0, 0, \cdots, 0, 1]$. Since $P_n(\mathbb{K})$ contains $GL_{n-1}(\mathbb{K})$, and since $P_n(\mathbb{K})$, $P_n(\mathbb{K})^t$ and the center $\mathbb{K}^{\times}$ generate the group $GL_n(\mathbb{K})$, we have the following

Corollary C. *Every generalized function on $GL_n(\mathbb{K})$ which is invariant under the adjoint action of $P_n(\mathbb{K})$ is invariant under the adjoint action of the whole group $GL_n(\mathbb{K})$.*

We remark that under the additional assumption that the generalized function is an eigenvector of the algebra of bi-invariant differential operators on $GL_n(\mathbb{K})$, Corollary C is the main result of [Ba03] (Theorem 1.4). As observed by Kirillov, this implies the validity of his famous conjecture on $GL_n(\mathbb{K})$, namely that every irreducible unitary representation of $GL_n(\mathbb{K})$ remains irreducible when restricted to the subgroup $P_n(\mathbb{K})$. We refer the readers to [Ba03] for details.

Here are some words on the approaches, contents and the organization of this paper. In Section 2, we examine the space of tempered generalized functions with support properties for both the functions and their Fourier transforms, as a module

for the Weyl algebra. A key result (Proposition 2.7) says that certain such modules are complete reducible with expected irreducible factors. In Section 3, we introduce the notion of a hermitian A-module, where A is a commutative involutive algebra over $\mathbb{R}$. Then the group G in this paper becomes the isometry group $\mathrm{U}(E)$ of a hermitian A-module E, corresponding to one of the five simple commutative involutive algebras A. We then prove in this context (Proposition 3.3) that a (weighted) Euler vector field acts semisimply on a certain space of tempered generalized functions on E, and all its eigenvalues are nonnegative integers. Note that the proof of this positivity result depends in a rather crucial way the rigidity assertions of Section 2. In Section 4, we introduce a group $\tilde{\mathrm{U}}(E)$ and an action of $\tilde{\mathrm{U}}(E)$ on $\mathfrak{u}(E) \times E$, where $\mathfrak{u}(E)$ is the Lie algebra of $\mathrm{U}(E)$. The group $\tilde{\mathrm{U}}(E)$ has a quadratic character χ_E with kernel $\mathrm{U}(E)$, and the key object of concern is then $C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$, the space of χ_E-equivariant tempered generalized functions on $\mathfrak{u}(E) \times E$. We prove in Proposition 4.1 a reduction result for such generalized functions within the null cone, by using metrically properness of nondistinguished nilpotent orbits, or by appealing to the eigenvalue estimate of Section 3 for distinguished nilpotent orbits. Sections 2, 3, and 4 are at the heart of our approaches.

In Section 5, we carry out the reduction to the null cone (Proposition 5.3) by a form of Harish-Chandra descent. We then see in Section 6 that results of Sections 4 and 5 allow us to conclude the vanishing of $C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$. This leads us to Theorem 6.5, which is a reformulation of Theorem B. In Section 7, we derive Theorem A from Theorem B by using a version of the Gelfand-Kazhdan criterion. Notwithstanding the fact that the general lines of the concluding three sections are known to the experts (see [GK75, Be84, JR96, AGRS] for related references), the approaches taken by the current article, in terms of hermitian A-modules, have some distinct advantages, at least for the problem at hand.

2. Rigidity of some generalized functions

Recall the space $C^{-\infty}(M)$ of generalized functions on a manifold M. For any locally closed subset Z of M, denote by

$$(1) \qquad\qquad C^{-\infty}(M; Z) \subset C^{-\infty}(U)$$

the subspace consisting of all f which are supported in Z, where U is an open subset of M containing Z as a closed subset. This definition is independent of U.

If M is a Nash manifold, denote by $C^{-\xi}(M) \subset C^{-\infty}(M)$ the space of tempered generalized functions on M. We refer the interested reader to [Sh, AG1] on generalities of Nash manifolds and tempered generalized functions. (For a short introduction, see [JSZ].) We say that a subset Z of a Nash manifold M is locally Nash closed if there is an open semialgebraic subset U of M, which contains Z as a closed semialgebraic

subset. In this case, denote by $C^{-\xi}(M; Z)$ the subspace of $C^{-\xi}(U)$ consisting of all f which are supported in Z. Again this is independent of U.

Let F be a finite dimensional real vector space, which is canonically a Nash manifold. Denote by $W[F]$ the space of all (complex) polynomial coefficient differential operators on F, called the Weyl algebra of F. It contains the algebra $\mathbb{C}[F]$ of all polynomial functions, and the algebra $D[F]$ of all constant coefficient differential operators. Furthermore, the multiplication map

$$D[F] \otimes \mathbb{C}[F] \rightarrow W[F]$$

is a vector space isomorphism.

The space $C^{-\xi}(F)$ is a $W[F]$-module in a natural way. Here is an example of irreducible $W[F]$-submodule of $C^{-\xi}(F)$ with a simple structure: $C^{-\xi}(F; \{0\})$. It has a distinguished nonzero element δ_F (called the Dirac function), which is characterized (up to a nonzero scalar) by the equation

$$\lambda \delta_F = 0, \quad \lambda \in F^*,$$

where F^* is the space of real valued linear functionals on F. More generally, we define the following analog of $C^{-\xi}(F; \{0\})$ for each subspace F' of F:

(2) $\qquad C^{-\xi}(F; F')^{\partial F'} := \{f \in C^{-\xi}(F; F') \mid \partial v \text{ is nilpotent on } f, v \in F'\}.$

Here and henceforth, $\partial v := \frac{\partial}{\partial v}$ is the partial derivative along v, and we say that a linear operator is nilpotent on a vector if some positive power of the linear operator annihilates the vector.

Lemma 2.1. *If $F = F' \oplus F''$ is a direct sum decomposition, then*

$$C^{-\xi}(F; F')^{\partial F'} = \mathbb{C}[F'] \otimes C^{-\xi}(F''; \{0\}),$$

and consequently it is an irreducible $W[F]$-module.

Proof. Note that every tempered generalized function has a finite order. Hence by the well-known result of L. Schwartz about local representation of a generalized function with support, we have

$$C^{-\xi}(F; F') = C^{-\xi}(F') \otimes C^{-\xi}(F''; \{0\}).$$

The lemma then follows easily. $\qquad \square$

The following lemma says that $C^{-\xi}(F; F')^{\partial F'}$ is typical within a certain category of $W[F]$-modules. It may be viewed as an algebraic version of the Stone-Von Neumann theorem. See [W93], Lemma 3 of Appendix 1.

Lemma 2.2. *Let $\mathcal{V}$ be a $W[F]$-module such that every $\lambda \in (F/F')^*$ and every ∂v ($v \in F'$) act locally nilpotently. Then $\mathcal{V}$ splits into a direct sum of irreducible $W[F]$-modules, each of which is isomorphic to $C^{-\xi}(F; F')^{\partial F'}$.*

From now on, we further assume that F is a non-degenerate real quadratic space, i.e., it is equipped with a non-degenerate symmetric bilinear form $\langle\,,\,\rangle_F$. Recall that the Fourier transform is the topological linear automorphism

$$\mathcal{F}_F : \mathcal{S}(F) \to \mathcal{S}(F)$$

of the Schwartz function space $\mathcal{S}(F)$, given by

$$(\mathcal{F}_F f)(x) = \int_F f(y) e^{-2\pi\sqrt{-1}\,\langle x,y\rangle_F}\,dy.$$

Here dy is the Lebesgue measure on F, normalized so that the volume of the cube

$$\{t_1 v_1 + t_2 v_2 + \cdots + c_r v_r \mid 0 \le t_1, t_2, \cdots, t_r \le 1\}$$

is 1, for any orthogonal basis $v_1, v_2, \cdots, v_r$ of F such that

$$\langle v_i, v_i\rangle_F = \pm 1, \quad i = 1, 2, \cdots, r.$$

Recall also that as a topological vector space, $C^{-\xi}(F)$ is the strong dual of the Fréchet space of Schwartz densities on F. It contains $\mathcal{S}(F)$ as a dense subspace, and the Fourier transform extends continuously to a topological linear isomorphism

$$\mathcal{F}_F : C^{-\xi}(F) \to C^{-\xi}(F),$$

which is still called the Fourier transform.

For any two closed semialgebraic subsets Z_1 and Z_2 of F, denote by

(3) $$C^{-\xi}(F; Z_1, Z_2) \subset C^{-\xi}(F)$$

the subspace consisting of all f such that

- f is supported in Z_1, and
- $\mathcal{F}_F(f)$ is supported in Z_2.

It is a $W[F]$-submodule of $C^{-\xi}(F)$. For the rest of the section, we will be concerned with the structure of such $W[F]$-submodules.

For a subspace F' of F, let $F'^\perp$ denote its perpendicular space:

$$F'^\perp := \{v \in F \mid \langle v, v'\rangle_F = 0,\ v' \in F'\}.$$

Note that the Fourier transform $\mathcal{F}_F(f)$ of $f \in C^{-\xi}(F)$ is supported in $F'^\perp$ if and only if for all $v \in F'$, ∂v is nilpotent on f. Therefore

(4) $$C^{-\xi}(F; F')^{\partial F'} = C^{-\xi}(F; F', F'^\perp).$$

For later use, we record the following

Proposition 2.3. *If F^0 is a non-degenerate subspace of F, and*

$$(F^0)^\perp = F^\oplus \oplus F^-$$

is a decomposition to totally isotropic subspaces, then

$$C^{-\xi}(F; F^+ \oplus F^0, F^+ \oplus F^0) = \mathbb{C}[F^+] \otimes C^{-\xi}(F^-; \{0\}) \otimes C^{-\xi}(F^0).$$

Proof. The proof is similar to that of Lemma 2.1. □

Following [JSZ], we make the following

Definition 2.4. *(a). A submanifold Z of F is said to be metrically proper if for every $z \in Z$, the tangent space $\mathrm{T}_z(Z)$ is contained in a proper non-degenerate subspace of F.*

(b). A locally closed subset Z of F is said to be piecewise metrically proper if there is a finite filtration

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_k = \emptyset$$

of Z by its closed subsets, so that every difference $Z_i \setminus Z_{i+1}$ is a metrically proper submanifold of F, $i = 0, 1, \cdots, k - 1$.

Denote by Δ_F the Laplacian operator on F. If $v_1, v_2, \cdots, v_r$ is a basis of F, and $v'_1, v'_2, \cdots, v'_r$ is the dual basis with respect to $\langle\,,\,\rangle_F$, then

$$(5) \qquad \Delta_F = \sum_{i=1}^{r} \partial v_i \, \partial v'_i.$$

The following is a special case of Lemma 3.2 in [JSZ].

Lemma 2.5. *Let F be a finite dimensional non-degenerate real quadratic space, and Z be a piecewise metrically proper locally closed subset of it. If $f \in C^{-\infty}(F; Z)$ is annihilated by some positive power of Δ_F, then $f = 0$.*

Remark: A tempered generalized function f on F is annihilated by some positive power of Δ_F if and only if its Fourier transform $\mathcal{F}_F(f)$ is supported in the null cone

$$\Gamma_F := \{v \in F \mid \langle v, v \rangle_F = 0\}.$$

Lemma 2.6. *Assume $\dim_{\mathbb{R}} F = 2r$. Let F^+ be a totaly isotropic subspace of F of dimension r. Let $\mathcal{V}$ be a $W[F]$-module on which Δ_F and every $\lambda \in (F/F^+)^*$ act locally nilpotently. Then every ∂v ($v \in F^+$) also acts locally nilpotently on $\mathcal{V}$. Consequently, $\mathcal{V}$ is generated by*

$$\{f \in \mathcal{V} \mid \lambda f = (\partial v)f = 0, \lambda \in (F/F^+)^*, v \in F^+\}.$$

Proof. The second assertion follows from the first, in view of Lemma 2.1 and Lemma 2.2.

To prove the first assertion, take a totaly isotropic subspace F^- of F which is complementary to F^+. Note that

$$W[F] = W[F^+] \otimes W[F^-].$$

View $\mathcal{V}$ as a $W[F^-]$-module and apply Lemma 2.2 to it, we have

(6) $$\mathcal{V} = \mathcal{V}' \otimes C^{-\xi}(F^-; \{0\}),$$

where

$$\mathcal{V}' = \mathrm{Hom}_{W[F^-]}(C^{-\xi}(F^-; \{0\}), \mathcal{V})$$

is canonically a $W[F^+]$-module, and (6) is an identification of $W[F]$-modules.

Take a basis $u_1, u_2, \cdots, u_r$ of F^+, and the dual basis $v_1, v_2, \cdots, v_r$ of F^-. Then

$$\Delta_F = 2 \sum_{i=1}^{r} \partial u_i \, \partial v_i.$$

Let $f' \in \mathcal{V}'$. Then for some positive integer m, we have

$$\Delta_F^m(f' \otimes \delta_{F^-}) = 0,$$

i.e.,

$$\sum_{\substack{i_1 + i_2 + \cdots + i_r = m, \\ i_1, i_2, \cdots, i_r \geq 0}} \binom{m}{i_1, i_2, \cdots, i_r} (\partial u_1)^{i_1} (\partial u_2)^{i_2} \cdots (\partial u_r)^{i_r} f'$$

$$\otimes (\partial v_1)^{i_1} (\partial v_2)^{i_2} \cdots (\partial v_r)^{i_r} \delta_{F^-} = 0.$$

Therefore

$$(\partial u_1)^{i_1} (\partial u_2)^{i_2} \cdots (\partial u_r)^{i_r} f' = 0,$$

which proves that $\partial u_1, \partial u_2, \cdots, \partial u_r$ act locally nilpotently on $\mathcal{V}'$, and the lemma follows. $\square$

We are now ready to prove the following

Proposition 2.7. *Assume* $\dim_{\mathbb{R}} F = 2r$. *Let* $F_1, F_2, \cdots, F_s$ *be a set of (distinct) totally isotropic subspaces of* F, *each of dimension* r. *Then the* $W[F]$-*module*

$$C^{-\xi}(F; F_1 \cup F_2 \cup \cdots \cup F_s, F_1 \cup F_2 \cup \cdots \cup F_s)$$

is completely reducible with finite length, and with each irreducible factor isomorphic to some $C^{-\xi}(F; F_i, F_i)$.

Remark: We expect that

$$C^{-\xi}(F; F_1 \cup F_2 \cup \cdots \cup F_s, F_1 \cup F_2 \cup \cdots \cup F_s) = \bigoplus_{i=1}^{s} C^{-\xi}(F; F_i, F_i).$$

Proposition 2.7 is nevertheless sufficient for our purpose.

Proof. For any nonempty open connected semialgebraic subset F° of a totaly isotropic subspace F^+ of F, of dimension r, set

$$(7) \qquad \mathcal{V}_{F,F^\circ} := \{f \in C^{-\xi}(F;F^\circ) \mid \Delta_F \text{ is nilpotent on } f\}.$$

Then we have the restriction map

$$(8) \qquad C^{-\xi}(F;F^+,F^+) \to \mathcal{V}_{F,F^\circ}.$$

We claim that it is a $W[F]$-module isomorphism.

Clearly the map (8) is a well-defined nonzero $W[F]$-module homomorphism. It is injective since $C^{-\xi}(F;F^+,F^+)$ is irreducible. The space $\mathcal{V}_{F,F^\circ}$ is also a $W[F]$-module satisfying the conditions of Lemma 2.6. Therefore, it is generated by

$$\{f \in C^{-\xi}(F;F^\circ) \mid \lambda f = (\partial v)f = 0, \ \lambda \in (F/F^+)^*, \ v \in F^+\}$$
$$= \{\text{constant function on } F^\circ\} \otimes \delta_{F^-},$$

where F^- is a totally isotropic subspace of F which is complementary to F^+. Consequently, the map (8) is surjective as well.

Set

$$\tilde{Z} := \bigcup F_i \quad \text{and} \quad Z := \bigcup_{i \neq j} (F_i \cap F_j).$$

Label the connected components of $\tilde{Z} \setminus Z$ by $F_1^\circ, F_2^\circ, \cdots, F_N^\circ$. Clearly each of them is contained in some F_i as an open semialgebraic subset. Since $\tilde{Z}$ is contained in the null cone, any $f \in C^{-\xi}(F;\tilde{Z},\tilde{Z})$ is annihilated by some positive power of Δ_F. Therefore the restrictions yield a $W[F]$-module homomorphism

$$(9) \qquad C^{-\xi}(F;\tilde{Z},\tilde{Z}) \to \prod_{k=1}^{N} \mathcal{V}_{F,F_k^\circ},$$

with kernel $C^{-\xi}(F;Z,\tilde{Z})$.

Define a filtration

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_r = \emptyset$$

of Z by

$$Z_k := \bigcup_{\substack{\dim F_i \cap F_j \leq r-1-k, \\ i \neq j}} (F_i \cap F_j).$$

Since every subspace of F of dimension $< r$ is metrically proper, we see from the filtration that Z is piecewise metrically proper in F. Now Lemma 2.5 implies that

$$C^{-\xi}(F;Z,\tilde{Z}) = 0.$$

Therefore the map in (9) is injective and we finish the proof by using the isomorphism in (8). □

3. Eigenvalue estimate of an Euler vector field

In this section, we first describe a general set-up in order to work with all five series of classical groups in a uniform manner. We then prove in this context an eigenvalue estimate of an Euler vector field acting on a certain space of tempered generalized functions.

Let A be a finite dimensional semi-simple commutative algebra over $\mathbb{R}$, which is thus a finite product of copies of $\mathbb{R}$ and $\mathbb{C}$. Let E be an A-module of finite dimension, i.e.,

$$\dim_A(E) := \max \left\{ \dim_{A_0}(A_0 \otimes_A E) \mid A_0 \text{ is a quotient field of } A \right\} < +\infty.$$

Denote by $\mathfrak{gl}_A(E)$ the A-algebra of A-endomorphisms of E, and by

$$\mathrm{tr}_A : \mathfrak{gl}_A(E) \to A$$

the trace map, which is specified by requiring that the diagram

$$
\begin{array}{ccc}
\mathfrak{gl}_A(E) & \xrightarrow{\ \mathrm{tr}_A\ } & A \\
{\scriptstyle 1_{A_0}\otimes}\Big\downarrow & & \Big\downarrow \\
\mathfrak{gl}_{A_0}(A_0 \otimes_A E) & \xrightarrow{\ \mathrm{tr}\ } & A_0
\end{array}
$$

commutes for every quotient field A_0 of A, where the bottom arrow is the usual trace map. Set

$$\mathfrak{sl}_A(E) := \{ x \in \mathfrak{gl}_A(E) \mid \mathrm{tr}_A(x) = 0 \}.$$

From now on, we assume that a $\mathbb{R}$-algebra involution τ on A is given. We call (A, τ) (or A when τ is understood) a commutative involutive algebra. The commutative involutive algebra A is said to be simple if it is nonzero, and has no τ-stable ideal except for $\{0\}$ and itself. Every simple commutative involutive algebra is isomorphic to one of the followings:

(10) $$(\mathbb{R}, 1),\ (\mathbb{C}, 1),\ (\mathbb{C}, ^{-}),\ (\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}}),\ (\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}}),$$

where $\tau_{\mathbb{R}}$ and $\tau_{\mathbb{C}}$ are the maps which interchange the coordinates. The first three cases will be referred to as Type I, and the last two cases as Type II.

A $\mathbb{R}$-bilinear map

$$\langle\,,\,\rangle_E : E \times E \to A$$

is called a hermitian form if it satisfies

$$\langle u, v \rangle_E = (\langle v, u \rangle_E)^{\tau}, \quad \langle au, v \rangle_E = a \langle u, v \rangle_E, \quad a \in A,\ u, v \in E.$$

We will always assume that E is a hermitian A-module, namely it is equipped with a non-degenerate hermitian form $\langle\,,\,\rangle_E$. Denote by $\mathrm{U}(E)$ the group of all A-module

automorphisms of E which preserve the form $\langle\,,\,\rangle_E$, and by $\mathfrak{u}(E)$ its Lie algebra, which consists of all $x \in \mathfrak{gl}_A(E)$ such that

$$\langle xu, v\rangle_E + \langle u, xv\rangle_E = 0, \quad u, v \in E.$$

Set

$$\mathrm{U}(A) := \{a \in A^\times \mid a^\tau a = 1\}.$$

Through scalar multiplication, there is a homomorphism

$$\mathrm{U}(A) \to \mathrm{U}(E),$$

whose image, which coincides with the center of $\mathrm{U}(E)$, is denoted by $\mathrm{Z}(E)$. Similarly, set

$$\mathfrak{u}(A) := \{a \in A \mid a^\tau + a = 0\},$$

and denote by $\mathfrak{z}(E)$ the image of the map (again through scalar multiplication)

$$\mathfrak{u}(A) \to \mathfrak{u}(E).$$

Then $\mathfrak{u}(A)$ is the Lie algebra of $\mathrm{U}(A)$, and $\mathfrak{z}(E)$ is the Lie algebra of $\mathrm{Z}(E)$. (Note that $\mathfrak{z}(E)$ may not coincide with the center of $\mathfrak{u}(E)$.) Set

$$\mathfrak{su}(E) := \mathfrak{u}(E) \cap \mathfrak{sl}_A(E).$$

Then

(11) $$\mathfrak{u}(E) = \mathfrak{z}(E) \oplus \mathfrak{su}(E).$$

When (A, τ) is one of the five simple commutative involutive algebras in (10), then accordingly, every hermitian A-module must be isomorphic to one of the followings:

(12) $$(\mathbb{R}^{p+q}, \langle\,,\,\rangle_{\mathrm{O}(p,q)}), \quad (\mathbb{C}^n, \langle\,,\,\rangle_{\mathrm{O}(n)}), \quad (\mathbb{C}^{p+q}, \langle\,,\,\rangle_{\mathrm{U}(p,q)}),$$
$$(\mathbb{R}^n \oplus \mathbb{R}^n, \langle\,,\,\rangle_{\mathbb{R},n}), \quad (\mathbb{C}^n \oplus \mathbb{C}^n, \langle\,,\,\rangle_{\mathbb{C},n}),$$

where $p, q, n \geq 0$, and all spaces involved are considered as spaces of column vectors. The corresponding hermitian forms are given as follows: $\langle\,,\,\rangle_{\mathrm{O}(p,q)}$ is the symmetric form defined by the matrix $I_{p,q}$, $\langle\,,\,\rangle_{\mathrm{O}(n)}$ is the standard symmetric form on $\mathbb{C}^n$, $\langle\,,\,\rangle_{\mathrm{U}(p,q)}$ is the usual hermitian form defined by the matrix $I_{p,q}$, $\langle\,,\,\rangle_{\mathbb{R},n}$ and $\langle\,,\,\rangle_{\mathbb{C},n}$ are the maps given by

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u' \\ v' \end{bmatrix} \right) \mapsto (v'^t u, u'^t v).$$

The group $\mathrm{U}(E)$ corresponding to (12) is isomorphic to one of the followings:

(13) $$\mathrm{O}(p, q), \ \mathrm{O}_n(\mathbb{C}), \ \mathrm{U}(p, q), \ \mathrm{GL}_n(\mathbb{R}), \ \mathrm{GL}_n(\mathbb{C}).$$

Assume in the rest of this section that A is simple. Fix an element c_A in $\mathfrak{u}(A)$ so that

(14)
$$c_A^2 = \begin{cases} 0, & \text{if } (A, \tau) \cong (\mathbb{K}, 1), \\ -1, & \text{if } (A, \tau) \cong (\mathbb{C}, ^-), \\ 1, & \text{if } (A, \tau) \cong (\mathbb{K} \times \mathbb{K}, \tau_{\mathbb{K}}), \end{cases}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, as before. Note that such a c_A is unique up to a sign.

For any $v \in E$, write
$$\phi_v(u) := \langle u, v \rangle_E\, v, \quad u \in E,$$
then $\phi_v \in \mathfrak{gl}_A(E)$. Denote by $\phi_v' \in \mathfrak{sl}_A(E)$ the projection of ϕ_v to the second factor according to the decomposition
$$\mathfrak{gl}_A(E) = \{\text{scalar multiplication}\} \oplus \mathfrak{sl}_A(E).$$

For any $x \in \mathfrak{su}(E)$, set

(15)
$$\psi_{x,v} := \begin{cases} c_A\, \phi_v', & \text{if } c_A \neq 0, \\ x\phi_v + \phi_v x, & \text{if } c_A = 0, \end{cases}$$

which is checked to be in $\mathfrak{su}(E)$. Following [AGRS], we define

(16)
$$E(x) := \{\, v \in E \mid \psi_{x,v} \in [\mathfrak{su}(E), x]\, \}.$$

For any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_A(E)$, denote by $\mathfrak{h}^x$ the centralizer of x in $\mathfrak{h}$. An element $x \in \mathfrak{su}(E)$ is said to be nilpotent if it is nilpotent as a $\mathbb{R}$-linear operator on E. The following lemma gives a description of $E(x)$.

Lemma 3.1. *Let* $x \in \mathfrak{su}(E)$.

(a) *If* $c_A \neq 0$, *then*
$$E(x) = \{\, v \in E \mid \langle yv, v \rangle_E = 0,\ y \in \mathfrak{sl}_A(E)^x\, \}.$$

(b) *If* $c_A = 0$, *then*
$$E(x) = \{\, v \in E \mid \langle xyv, v \rangle_E = 0,\ y \in \mathfrak{gl}_A(E)^x\, \}.$$

(c) *In all cases, if* x *is nilpotent, then*
$$\langle x^i v, v \rangle_E = 0 \quad \text{for all } v \in E(x) \text{ and } i > 0.$$

Proof. We only prove Part (a). Part (b) is proved similarly, and Part (c) follows obviously from (a) and (b). So we assume that $c_A \neq 0$ and let $v \in E$. For simplicity, we write $\psi_{x,v} := c_A \phi_v'$ as ψ_v.

If $v \in E(x)$, then $\psi_v = [z, x]$ for some $z \in \mathfrak{su}(E)$. Therefore, for all $y \in \mathfrak{sl}_A(E)^x$,
$$\langle yv, v \rangle_E = \mathrm{tr}_A(\phi_v y) = \mathrm{tr}_A(\phi_v' y)$$
$$= (c_A)^{-1}\, \mathrm{tr}_A(\psi_v\, y) = (c_A)^{-1}\, \mathrm{tr}_A(zxy - xzy)$$
$$= (c_A)^{-1}\, \mathrm{tr}_A(zyx - xzy) = 0.$$

On the other hand, assume that for all $y \in \mathfrak{sl}_A(E)^x$, we have

$$\langle yv, v \rangle_E = 0, \quad i.e., \quad \mathrm{tr}_A(\psi_v\, y) = 0.$$

In particular, we have

$$\psi_v \in \{\, z \in \mathfrak{su}(E) \mid \mathrm{tr}_A(zy) = 0,\, y \in \mathfrak{su}(E)^x \,\}.$$

It is easy to see that the latter space is precisely $[\mathfrak{su}(E), x]$ (c.f. [CM93, Page 14]). This finishes the proof.

$\square$

Denote by

(17) $$\Gamma_E := \{v \in E \mid \langle v, v \rangle_E = 0\}$$

the null cone of E. View E as a real quadratic space by the form

$$\langle u, v \rangle_{E,\mathbb{R}} := \mathrm{tr}_{A/\mathbb{R}}(\langle u, v \rangle_E),$$

where $\mathrm{tr}_{A/\mathbb{R}} : A \to \mathbb{R}$ is the usual trace map for commutative algebras.

For any finite dimensional real vector space F and any $x \in \mathrm{End}_{\mathbb{R}}(F)$, denote by

(18) $$\epsilon_{F,x} \in W[F]$$

the vector field on F whose tangent vector at $v \in F$ is xv. When $x = 1$ is the identity operator, this is the usual Euler vector field $\epsilon_F := \epsilon_{F,1}$.

For a nilpotent element $\mathbf{e} \in \mathfrak{su}(E)$, define

(19) $$\mathcal{V}_{E,\mathbf{e}} := C^{-\xi}(E; E(\mathbf{e}) \cap \Gamma_E, E(\mathbf{e}) \cap \Gamma_E)^{\mathrm{Z}(E)},$$

where and as usual, a superscript group indicates the group invariants. Clearly

(20) $$\epsilon_{E,c_A} f = 0, \quad f \in \mathcal{V}_{E,\mathbf{e}}.$$

The space $\mathcal{V}_{E,\mathbf{e}}$ arises naturally when one carries out the reduction within the null cone. See Lemma 4.7.

Proposition 3.2. *Assume that* $\dim_A(E) = 1$.

(a) *If A is of Type I, then*

$$C^{-\xi}(E; \Gamma_E, \Gamma_E) = \{0\}.$$

(b) *If A is of Type II, then for every $f \in C^{-\xi}(E; \Gamma_E, \Gamma_E)$,*

$$\epsilon_{E,c_A} f = 0 \quad implies \quad f = 0.$$

Consequently, in all cases, $\mathcal{V}_{E,\mathbf{e}} = \{0\}$ for the only element $\mathbf{e} \in \mathfrak{su}(E) = \{0\}$.

Proof. In case (a), we have $\Gamma_E = \{0\}$, which is metrically proper in E. Therefore the lemma follows from Lemma 2.5.

In case (b), we assume that

$$(E, \langle\,,\,\rangle_E) = (\mathbb{K} \oplus \mathbb{K}, \langle\,,\,\rangle_{\mathbb{K},1})$$

as in (12). Then

$$\Gamma_E = F_0 \cup F_1$$

is the union of two totally isotropic subspaces F_0 and F_1, where

$$F_0 := \{0\} \oplus \mathbb{K}, \quad \text{and} \quad F_1 := \mathbb{K} \oplus \{0\}.$$

By Proposition 2.7, it suffices to show that for every $f \in C^{-\xi}(E; F_i, F_i)$,

$$\epsilon_{E,c_A} f = 0 \quad \text{implies} \quad f = 0.$$

To fix the sign, assume that $c_A = (1, -1)$. By Lemma 2.1,

$$C^{-\xi}(E; F_0, F_0) = \mathbb{C}[F_0] \otimes C^{-\xi}(F_1; \{0\}),$$

and therefore ϵ_{E,c_A} acts semisimply on it, and all its eigenvalues are negative integers. Likewise, ϵ_{E,c_A} acts semisimply on $C^{-\xi}(E; F_1, F_1)$, and all its eigenvalues are positive integers. This finishes the proof. $\square$

Recall that a nilpotent element $\mathbf{e} \in \mathfrak{su}(E)$ is said to be distinguished if it commutes with no nonzero semisimple element in $\mathfrak{su}(E)$ (c.f. [CM93, Section 8.2]).

The rest of this section is devoted to a proof of the following

Proposition 3.3. *Let* $\mathbf{h}, \mathbf{e}, \mathbf{f} \in \mathfrak{su}(E)$ *be a standard triple, i.e.,*

$$[\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, \quad [\mathbf{h}, \mathbf{f}] = 2\mathbf{f}, \quad [\mathbf{e}, \mathbf{f}] = \mathbf{h}.$$

Assume that $\dim_A(E) \geq 2$ *and* $\mathbf{e}$ *is distinguished. Then the vector field* $\epsilon_{E,\mathbf{h}}$ *acts semisimply on* $\mathcal{V}_{E,\mathbf{e}}$, *and all its eigenvalues are nonnegative integers.*

For every $\mathbf{h} \in \mathfrak{su}(E)$ in a standard triple $\mathbf{h}, \mathbf{e}, \mathbf{f}$, denote by $E_{\mathbf{h}}^i \subset E$ the eigenspace of $\mathbf{h}$ with eigenvalue i, where $i \in \mathbb{Z}$. Write

$$E_{\mathbf{h}}^+ := \bigoplus_{i>0} E_{\mathbf{h}}^i, \quad \text{and} \quad E_{\mathbf{h}}^- := \bigoplus_{i<0} E_{\mathbf{h}}^i.$$

Next we prove (a stronger version of) Proposition 3.3 when A is of Type I.

Lemma 3.4. *Assume that A is of Type I and $\dim_A(E) \geq 2$. Let $\mathbf{h}, \mathbf{e}, \mathbf{f} \in \mathfrak{su}(E)$ be a standard triple, where $\mathbf{e}$ is a distinguished nilpotent element in $\mathfrak{su}(E)$. Then*

(a) $E(\mathbf{e})$ *is contained in* $E_{\mathbf{h}}^+ + E_{\mathbf{h}}^0$;

(b) *the vector field* $\epsilon_{E,\mathbf{h}}$ *acts semisimply on* $C^{-\xi}(E; E(\mathbf{e}), E(\mathbf{e}))$, *and all its eigenvalues are nonnegative integers.*

Proof. As usual, view E as a $\mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} A$-modules via the standard triple. Let

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_s \tag{21}$$

be a decomposition of E into irreducible $\mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} A$-modules. By the classification of distinguished nilpotent orbits ([CM93, Theorem 8.2.14]), we know that $s = 1$ if $(A, \tau) = (\mathbb{C}, ^-)$. If $(A, \tau) = (\mathbb{K}, 1)$, then (21) is an orthogonal decomposition, and $E_1, E_2, \cdots, E_s$ have pairwise different odd dimensions.

Suppose that we are in the latter (orthogonal) case. Denote by $\mathbf{e}_i \in \mathfrak{su}(E_i)$ the restriction of $\mathbf{e}$ to E_i. It is easy to see that ([AGRS, Lemma 5.3])

$$E(\mathbf{e}) \subset E_1(\mathbf{e}_1) + E_2(\mathbf{e}_2) + \cdots + E_r(\mathbf{e}_r).$$

To show Part (a), we may therefore assume that E is irreducible as a $\mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} A$-module. Let $r \geq 0$ be its highest weight and

$$\{v_i \mid i = -r, -r+2, \cdots, r\}$$

be an A-basis of E such that

- v_i is an eigenvector of $\mathbf{h}$ with eigenvalue i, and
- $\mathbf{e}v_i = v_{i+2}, \quad i < r.$

Assume that there is an element

$$v = a_{-r}v_{-r} + a_{-r+2}v_{-r+2} + \cdots + a_r v_r \in E(\mathbf{e}) \setminus (E_{\mathbf{h}}^+ + E_{\mathbf{h}}^0).$$

Denote by $j > 0$ the largest number so that $a_{-j} \neq 0$. Then

$$\langle \mathbf{e}^j v, v \rangle_E = a_{-j} \, a_{-j}^\tau \, \langle v_j, v_{-j} \rangle_E \neq 0,$$

which contradicts Part (c) of Lemma 3.1. This proves Part (a).

By Part (a) and Proposition 2.3, we have

$$\begin{aligned}
&C^{-\xi}(E; E(\mathbf{e}), E(\mathbf{e})) \\
&\subset C^{-\xi}(E; E_{\mathbf{h}}^+ + E_{\mathbf{h}}^0, E_{\mathbf{h}}^+ + E_{\mathbf{h}}^0) \\
&= \mathbb{C}[E_{\mathbf{h}}^+] \otimes C^{-\xi}(E_{\mathbf{h}}^-; \{0\}) \otimes C^{-\xi}(E_{\mathbf{h}}^0).
\end{aligned}$$

Part (b) therefore follows. $\qquad\square$

We are now left with the task of proving Proposition 3.3, when A is of Type II. Thus let $(A, \tau) = (\mathbb{K} \times \mathbb{K}, \tau_{\mathbb{K}})$, and

$$(E, \langle\,,\,\rangle_E) = (\mathbb{K}^n \oplus \mathbb{K}^n, \langle\,,\,\rangle_{\mathbb{K}, n})$$

be as in (12), with $n \geq 2$. Then

$$U(E) = \left\{ \begin{bmatrix} g & 0 \\ 0 & g^{-t} \end{bmatrix} \mid g \in \mathrm{GL}_n(\mathbb{K}) \right\} = \mathrm{GL}_n(\mathbb{K}),$$

and

$$\mathfrak{u}(E) = \left\{ \begin{bmatrix} x & 0 \\ 0 & -x^t \end{bmatrix} \mid x \in \mathfrak{gl}_n(\mathbb{K}) \right\} = \mathfrak{gl}_n(\mathbb{K}).$$

A distinguished nilpotent element $\mathbf{e}$ of $\mathfrak{su}(E)$ is principle and we may assume without loss of generality that

$$\mathbf{e} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{h} = \mathrm{diag}(n-1, n-3, \cdots, 3-n, 1-n).$$

Then it is easy to check that

$$E(\mathbf{e}) \cap \Gamma_E = F_0 \cup F_1 \cup \cdots \cup F_n,$$

where

$$F_i = (\mathbb{K}^i \oplus \{0\}^{n-i}) \oplus (\{0\}^i \oplus \mathbb{K}^{n-i}).$$

In view of Proposition 2.7 and (20), it suffices to prove the following

Lemma 3.5. *With notation as above, the vector field $\epsilon_{E,\mathbf{h}}$ acts semisimply on*

$$(22) \qquad\qquad \{f \in C^{-\xi}(E; F_i, F_i) \mid \epsilon_{E,c_A} f = 0\},$$

with all its eigenvalues nonnegative integers.

Proof. We prove the lemma for $\mathbb{K} = \mathbb{R}$. The complex case is proved in the same way.

Denote by $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ the standard coordinates of $\mathbb{R}^n \oplus \mathbb{R}^n$, and write

$$\partial_j = \frac{\partial}{\partial x_j} \quad \text{and} \quad d_j = \frac{\partial}{\partial y_j}, \qquad j = 1, 2, \cdots, n.$$

By Lemma 2.1, the space $C^{-\xi}(E; F_i, F_i)$ has a basis consisting of generalized functions of the form

$$f = x_1^{a_1} x_2^{a_2} \cdots x_i^{a_i} \, y_{i+1}^{b_{i+1}} y_{i+2}^{b_{i+2}} \cdots y_n^{b_n}$$
$$\otimes \partial_{i+1}^{a_{i+1}-1} \partial_{i+2}^{a_{i+2}-1} \cdots \partial_n^{a_n-1} \, d_1^{b_1-1} d_2^{b_2-1} \cdots d_i^{b_i-1} \delta_{F_i'},$$

where $a_1, ..., a_i, b_{i+1}, ..., b_n$ are nonnegative integers, and the rest of a's and b's are positive integers. Here and as before, $\delta_{F_i'}$ is a fixed Dirac function on the space

$$F_i' := (\{0\}^i \oplus \mathbb{K}^{n-i}) \oplus (\mathbb{K}^i \oplus \{0\}^{n-i}).$$

The generalized function f as above is an eigenvector for both ϵ_{E,c_A} and $\epsilon_{E,\mathbf{h}}$. The condition

$$\epsilon_{E,c_A} f = 0$$

amounts to

$$\sum_{j \leq i}(a_j + b_j) = \sum_{j > i}(a_j + b_j).$$

Then the $\epsilon_{E,\mathbf{h}}$-eigenvalue of f is

$$
\begin{aligned}
& (n-1)a_1 + (n-3)a_2 + \cdots + (n-2i+1)a_i \\
& -(n-2i-1)a_{i+1} - (n-2i-3)a_{i+2} - \cdots - (1-n)a_n \\
& +(n-1)b_1 + (n-3)b_2 + \cdots + (n-2i+1)b_i \\
& -(n-2i-1)b_{i+1} - (n-2i-3)b_{i+2} - \cdots - (1-n)b_n \\
& \geq (n-2i)a_1 + (n-2i)a_2 + \cdots + (n-2i)a_i \\
& -(n-2i)a_{i+1} - (n-2i)a_{i+2} - \cdots - (n-2i)a_n \\
& +(n-2i)b_1 + (n-2i)b_2 + \cdots + (n-2i)b_i \\
& -(n-2i)b_{i+1} - (n-2i)b_{i+2} - \cdots - (n-2i)b_n \\
& = 0.
\end{aligned}
$$

$\square$

Note that for $i = 0, n$, the space (22) is equal to zero.

4. REDUCTION WITHIN THE NULL CONE

Recall that we are given a commutative involutive algebra (A, τ) and a hermitian A-module E. Denote by $\tilde{U}(E)$ the subgroup of $\mathrm{GL}_{\mathbb{R}}(E) \times \{\pm 1\}$ consisting of pairs (g, δ) such that either

$$\delta = 1 \quad \text{and} \quad g \in \mathrm{U}(E),$$

or

(23)
$$
\begin{cases}
\delta = -1, \\
g(av) = a^\tau g(v), & a \in A,\ v \in E, \quad \text{and} \\
\langle gu, gv \rangle_E = \langle v, u \rangle_E, & u, v \in E.
\end{cases}
$$

Denote by

(24)
$$\chi_E : \tilde{U}(E) \to \{\pm 1\}$$

the quadratic character of $\tilde{U}(E)$ projecting to the second factor, which is easily checked to be surjective. Therefore, we get an exact sequence

$$\{1\} \to \mathrm{U}(E) \to \tilde{U}(E) \xrightarrow{\chi_E} \{\pm 1\} \to \{1\}.$$

Now let $\tilde{\mathrm{U}}(E)$ act on $\mathrm{U}(E)$ by

(25) $$(g, \delta)x := gx^\delta g^{-1},$$

and on E by

$$(g, \delta)v := \delta gv.$$

Let $\tilde{\mathrm{U}}(E)$ act on $\mathfrak{u}(E)$ through the differential at the identity of $\mathrm{U}(E)$, i.e.,

$$(g, \delta)x := \delta gxg^{-1}.$$

Let $\tilde{\mathrm{U}}(E)$ act on $\mathrm{U}(E) \times E$ and $\mathfrak{u}(E) \times E$ diagonally.

We introduce the following general notation. If H is a Lie group acting smoothly on a manifold M, then for any character χ_H of H, denote by

(26) $$C_{\chi_H}^{-\infty}(M) \subset C^{-\infty}(M)$$

the subspace consisting of all f which are χ_H-equivariant, i.e.,

$$f(hx) = \chi_H(h)f(x), \quad \text{for all } h \in H.$$

Similar notations (such as $C_{\chi_H}^{-\xi}(M; Z)$) apply without further explanation. We will be concerned with the space $C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E)$.

Denote by $\mathcal{N}_E \subset \mathfrak{su}(E)$ the null cone, which consists of all nilpotent elements in $\mathfrak{su}(E)$. Let

$$\mathcal{N}_E = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_k = \{0\} \supset \mathcal{N}_{k+1} = \emptyset$$

be a filtration of $\mathcal{N}_E$ by its closed subsets so that each difference

$$\mathcal{O}_i := \mathcal{N}_i \setminus \mathcal{N}_{i+1}, \quad 0 \leq i \leq k,$$

is an $\mathrm{U}(E)$-adjoint orbit.

Our aim is to prove the following reduction result for $C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E)$. Recall the null cone Γ_E of E.

Proposition 4.1. *Assume that A is simple, $\dim_A(E) \geq 1$, and that every element of $C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_i) \times \Gamma_E$, for some fixed $0 \leq i \leq k$. Then every element of $C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_{i+1}) \times \Gamma_E$.*

Note that $\mathfrak{u}(E) = \mathfrak{z}(E) \oplus \mathfrak{su}(E)$ is a $\tilde{\mathrm{U}}(E)$ stable decomposition, and $\tilde{\mathrm{U}}(E)$ acts on $\mathfrak{z}(E)$ trivially. Therefore by the localization principle (See [JSZ, Lemma 4.1], for example), for any fixed i,

every element of $C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_i) \times \Gamma_E$

$\iff$ every element of $C_{\chi_E}^{-\xi}(\mathfrak{su}(E) \times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$.

Thus it suffices to prove the following equivalent

Proposition 4.2. *Assume that A is simple, $\dim_A(E) \geq 1$, and that every element of $C_{\chi_E}^{-\xi}(\mathfrak{su}(E) \times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$, for some fixed $0 \leq i \leq k$. Then every element of $C_{\chi_E}^{-\xi}(\mathfrak{su}(E) \times E)$ is supported in $\mathcal{N}_{i+1} \times \Gamma_E$.*

For the ease of notation, denote

$$\mathfrak{s} := \mathfrak{su}(E).$$

We shall view $\mathfrak{s}$ as a non-degenerate real quadratic space via the form

$$\langle x, y \rangle_{\mathfrak{s}, \mathbb{R}} := \mathrm{tr}_{A/\mathbb{R}}(\mathrm{tr}_A(xy)).$$

Note that the null cone $\mathcal{N}_E$ is contained in the null cone of $\mathfrak{s}$ as a real quadratic space.

Lemma 4.3. *Let $\mathcal{O} \subset \mathcal{N}_E$ be a nilpotent $\mathrm{U}(E)$-orbit which is not distinguished. Then $\mathcal{O}$ is metrically proper in $\mathfrak{s}$.*

Proof. Let $x \in \mathcal{O}$. By definition, it commutes with a nonzero semisimple element $h \in \mathfrak{s}$. Denote by $\mathfrak{a}_h$ the center of $\mathfrak{s}^h$, which is a nonzero non-degenerate subspace of $\mathfrak{s}$.

Using the fact that every element of $\mathfrak{a}_h$ commutes with x, we see that the tangent space

$$\mathrm{T}_x(\mathcal{O}) = [\mathfrak{u}(E), x]$$

is contained in the proper non-degenerate subspace

$$(\mathfrak{a}_h)^{\perp} := \{y \in \mathfrak{s} \mid \langle y, z \rangle_{\mathfrak{s}, \mathbb{R}} = 0, \ z \in \mathfrak{a}_h\} \subset \mathfrak{s}.$$

$\square$

Lemma 4.4. *Proposition 4.2 holds when $\mathcal{O}_i$ is not distinguished.*

Proof. Let $f \in C_{\chi_E}^{-\xi}(\mathfrak{s} \times E)$. Then $\mathcal{F}_{\mathfrak{s}}(f) \in C_{\chi_E}^{-\xi}(\mathfrak{s} \times E)$, where $\mathcal{F}_{\mathfrak{s}}$ is the partial Fourier transform (along $\mathfrak{s}$) specified by the commutative diagram

$$
\begin{array}{ccc}
C^{-\xi}(\mathfrak{s} \times E) & = & C^{-\xi}(\mathfrak{s}) \widehat{\otimes} C^{-\xi}(E) \\
\downarrow{\scriptstyle \mathcal{F}_{\mathfrak{s}}} & & \downarrow{\scriptstyle \mathcal{F}_{\mathfrak{s}} \otimes 1} \\
C^{-\xi}(\mathfrak{s} \times E) & = & C^{-\xi}(\mathfrak{s}) \widehat{\otimes} C^{-\xi}(E) \,.
\end{array}
$$

By the assumption, the support of $\mathcal{F}_{\mathfrak{s}}(f)$ is contained in

$$\mathcal{N}_i \times \Gamma_E \subset \text{(the null cone of the real quadratic space } \mathfrak{s}) \times E.$$

Therefore, some positive power of the partial Laplacian $\Delta_{\mathfrak{s}}$ annihilates f. Now the lemma follows from Lemma 4.3 and (a variation of) Lemma 2.5. $\square$

Before proceeding further, we introduce a version of pull back of generalized functions.

Definition 4.5. *Let Z and Z' be locally closed subsets of manifolds M and M', respectively. A smooth map $\phi : M \to M'$ is said to be submersive from Z to Z' if*

- *ϕ is submersive at every point of Z, and*
- *for every $z \in Z$, there is an open neighborhood U of z in M such that*

$$\phi^{-1}(Z') \cap U = Z \cap U.$$

The following lemma is elementary.

Lemma 4.6. *If $\phi : M \to M'$ is submersive from Z to Z', as in Definition 4.5, then there is a unique linear map*

(27) $$\phi^* : C^{-\infty}(M'; Z') \to C^{-\infty}(M; Z),$$

with the following property: for any open subset U of M and U' of M', if

- *ϕ restricts to a submersive map $\phi_U : U \to U'$,*
- *$Z' \cap U'$ is closed in U', and*
- *$\phi_U^{-1}(Z' \cap U') = Z \cap U$,*

then the diagram

$$
\begin{array}{ccc}
C^{-\infty}(M'; Z') & \xrightarrow{\phi^*} & C^{-\infty}(M; Z) \\
\downarrow & & \downarrow \\
C^{-\infty}(U') & \xrightarrow{\phi_U^*} & C^{-\infty}(U)
\end{array}
$$

commutes, where the two vertical arrows are restrictions, and the bottom arrow is the usual pull back map of generalized functions via a submersion.

See [W88, Lemma 8.A.2.5] for the definition and properties of the usual pull back map. Note that the vertical arrows are well defined since $Z' \cap U'$ is closed in U', and $Z \cap U$ is closed in U. The map ϕ^* in (27) is still called the pull back. It is injective if $\phi(Z) = Z'$. In this case, we say that ϕ is submersive from Z onto Z'.

We continue the proof of Proposition 4.2. Recall the notations from Section 3.

Lemma 4.7. *Under the assumption of Proposition 4.2, the support of every $f \in C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$ is contained in*

$$(\mathcal{N}_{i+1} \times \Gamma_E) \cup \left(\bigsqcup_{\mathbf{e} \in \mathcal{O}_i} \{\mathbf{e}\} \times (E(\mathbf{e}) \cap \Gamma_E) \right).$$

Proof. We follow the method of [AGRS]. For every $t \in \mathbb{R}$, define a map

$$\eta := \eta_t : \mathfrak{s} \times E \quad \to \quad \mathfrak{s} \times E,$$
$$(x, v) \quad \mapsto \quad (x - t\psi_{x,v}, v),$$

which is checked to be submersive from $\mathfrak{s} \times \Gamma_E$ to $\mathfrak{s} \times \Gamma_E$. Therefore, by Lemma 4.6, it yields a pull back map

$$\eta^* : C^{-\infty}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E) \quad \to \quad C^{-\infty}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E).$$

Fix $f \in C_{\chi_E}^{-\xi}(\mathfrak{s} \times E)$. By our assumption, its support is contained in $\mathcal{N}_i \times \Gamma_E$, and so

$$f \in C_{\chi_E}^{-\xi}(\mathfrak{s} \times E; \mathcal{N}_i \times \Gamma_E) \subset C_{\chi_E}^{-\xi}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E).$$

Since the map η is algebraic and $\tilde{\mathrm{U}}(E)$-equivariant,

$$\eta^*(f) \in C_{\chi_E}^{-\xi}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E).$$

It is routine to check that η restricts to a bijection from $\mathfrak{s} \times \Gamma_E$ onto itself. Let $(\mathbf{e}, v) \in \mathcal{O}_i \times \Gamma_E$ be a point in the support of f. Denote by

$$\mathbf{e}' := \mathbf{e}'(\mathbf{e}, v, t) \in \mathfrak{s}$$

the unique element so that

$$\eta(\mathbf{e}', v) = (\mathbf{e}, v).$$

Then $(\mathbf{e}', v)$ is in the support of $\eta^*(f)$, and therefore our assumption implies that

(28) $$\mathbf{e}' \in \mathcal{N}_i.$$

A easy calculation shows that

$$\mathbf{e}' = \begin{cases} \mathbf{e} + t\psi_{\mathbf{e},v}, & \text{if } c_A \neq 0, \\ \mathbf{e} + t\psi_{\mathbf{e},v} + t^2 \phi_v \mathbf{e} \phi_v, & \text{if } c_A = 0. \end{cases}$$

Since $\mathcal{O}_i$ is open in $\mathcal{N}_i$, (28) implies that

$$\psi_{\mathbf{e},v} = \frac{d}{dt}|_{t=0} \mathbf{e}'(\mathbf{e}, v, t) \in \mathrm{T}_{\mathbf{e}}(\mathcal{O}_i) = [\mathfrak{u}(E), \mathbf{e}] = [\mathfrak{su}(E), \mathbf{e}],$$

i.e., $v \in E(\mathbf{e})$, and the proof is now complete. $\qquad\square$

For a nilpotent $\mathrm{U}(E)$-orbit $\mathcal{O} \subset \mathcal{N}_E$, denote by

(29) $$\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}} \subset C^{-\xi}(\mathfrak{s} \times E; \mathcal{O} \times E)^{\mathrm{U}(E)}$$

the subspace consisting of all f such that the supports of both f and its partial Fourier transform $\mathcal{F}_E(f)$ are contained in $\bigsqcup_{\mathbf{e} \in \mathcal{O}} \{\mathbf{e}\} \times (E(\mathbf{e}) \cap \Gamma_E)$.

Lemma 4.8. *Assume that A is simple, $\dim_A(E) \geq 1$, and $\mathcal{O}$ is distinguished. Then the Euler vector field $\epsilon_{\mathfrak{s}}$ acts semisimply on $\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}}$, and all its eigenvalues are real numbers $< -\frac{1}{2} \dim_{\mathbb{R}} \mathfrak{s}$.*

Let us prove the following

Lemma 4.9. *Lemma 4.8 implies Proposition 4.2 when $\mathcal{O}_i$ is distinguished.*

Proof. Denote by $q_{\mathfrak{s}}$ the quadratic form on $\mathfrak{s}$, i.e.,

$$q_{\mathfrak{s}}(x) = \langle x, x \rangle_{\mathfrak{s}, \mathbb{R}} = \mathrm{tr}_{A/\mathbb{R}}(\mathrm{tr}_A(x^2)).$$

The operators

$$\epsilon_{\mathfrak{s}} + \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{s}, \quad -\frac{1}{2} q_{\mathfrak{s}}, \quad \frac{1}{2} \Delta_{\mathfrak{s}}$$

form a standard triple in $W[\mathfrak{s}]$, and each of them leaves the space $\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i}$ stable. Lemma 4.8 says that $\epsilon_{\mathfrak{s}} + \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{s}$ is semisimple and has negative eigenvalues on $\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i}$, and so by [W88, Lemma 8.A.5.1], the map

$$\Delta_{\mathfrak{s}} : \mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i} \to \mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i}$$

is injective.

Let $f \in C_{\chi_E}^{-\xi}(\mathfrak{s} \times E)$. Applying Lemma 4.7 to f and its partial Fourier transform $\mathcal{F}_E(f)$, we conclude that under the restriction map

$$r_{\mathfrak{s} \times E} : C_{\chi_E}^{-\xi}(\mathfrak{s} \times E) \to C^{-\xi}(\mathfrak{s} \times E; \mathcal{O}_i \times E),$$

the image

$$r_{\mathfrak{s} \times E}(f) \in \mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i}.$$

Since the partial Fourier transform $\mathcal{F}_{\mathfrak{s}}(f)$ is again in $C_{\chi_E}^{-\xi}(\mathfrak{s} \times E)$, we see that $\mathcal{F}_{\mathfrak{s}}(f)$ is supported in

$$\mathcal{N}_i \times \Gamma_E \subset \text{(the null cone of the real quadratic space } \mathfrak{s}) \times E,$$

which implies that $r_{\mathfrak{s} \times E}(f)$ is annihilated by some positive power of $\Delta_{\mathfrak{s}}$. By the injectivity of $\Delta_{\mathfrak{s}}$ on $\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i}$, we conclude that $r_{\mathfrak{s} \times E}(f) = 0$ and we are done. $\square$

The remaining part of this section is devoted to the proof of Lemma 4.8.

For the moment, assume that $\mathcal{O} \subset \mathcal{N}_E$ is any nilpotent $\mathrm{U}(E)$-orbit (not necessarily distinguished). Pick any element $\mathbf{e} \in \mathcal{O}$ and extend it to a standard triple $\mathbf{h}, \mathbf{e}, \mathbf{f} \in \mathfrak{s}$. Then we have a vector space decomposition

$$\mathfrak{s} = [\mathfrak{s}, \mathbf{e}] \oplus \mathfrak{s}^{\mathbf{f}}.$$

Let $\mathrm{U}(E)$ act on $\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E$ via the left translation on the first factor. Define a $\mathrm{U}(E)$-equivariant map

$$(30) \qquad \begin{aligned} \theta : \mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E &\to \mathfrak{s} \times E, \\ (g, x, v) &\mapsto g(x + \mathbf{e}, v). \end{aligned}$$

Lemma 4.10. *The vector field*

$$\iota_{\mathbf{h}/2} + \epsilon_{\mathfrak{s}^{\mathbf{f}}, 1-\mathrm{ad}(\mathbf{h}/2)} - \epsilon_{E, \mathbf{h}/2} \tag{31}$$

on $\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E$ *is* θ*-related to the Euler vector field* $\epsilon_{\mathfrak{s}}$ *on* $\mathfrak{s} \times E$*, where* $\iota_{\mathbf{h}/2}$ *is the left invariant vector field on* $\mathrm{U}(E)$ *whose tangent vector at the identity is* $\mathbf{h}/2$.

Proof. Since both vector fields under consideration are $\mathrm{U}(E)$-invariant, it suffices to prove the θ-relatedness at a point of the form

$$\mathbf{x} := (1, x, v) \in \mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E.$$

Applying the differential of θ at $\mathbf{x}$, we have

$$
\begin{aligned}
\iota_{\mathbf{h}/2}|_{\mathbf{x}} = (\mathbf{h}/2, 0, 0) &\mapsto ([\mathbf{h}/2, x + \mathbf{e}], (\mathbf{h}/2)v), \\
\epsilon_{\mathfrak{s}^{\mathbf{f}}, 1-\mathrm{ad}(\mathbf{h}/2)}|_{\mathbf{x}} = (0, x - [\mathbf{h}/2, x], 0) &\mapsto (x - [\mathbf{h}/2, x], 0), \\
\epsilon_{E, \mathbf{h}/2}|_{\mathbf{x}} = (0, 0, (\mathbf{h}/2)v) &\mapsto (0, (\mathbf{h}/2)v).
\end{aligned}
$$

This implies the lemma since $\epsilon_{\mathfrak{s}}|_{\theta(\mathbf{x})} = (x + \mathbf{e}, 0)$. $\qquad\square$

Let $\mathrm{Z}(E)$ act on $\mathfrak{s}^{\mathbf{f}} \times E$ and $\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E$ via its action on the factor E. Then the map θ is $\mathrm{Z}(E)$-equivariant as well. Note that θ is submersive from $\mathrm{U}(E) \times \{0\} \times E$ onto $\mathcal{O} \times E$ (c.f. [W88, Page 299]). Therefore it yields an injective pull back map

$$C^{-\xi}(\mathfrak{s} \times E; \mathcal{O} \times E)^{\mathrm{U}(E)}$$
$$\xrightarrow{\theta^*} C^{-\xi}(\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E; \mathrm{U}(E) \times \{0\} \times E)^{\mathrm{U}(E) \times \mathrm{Z}(E)}.$$

Denote by

$$C^{-\xi}(\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E; \mathrm{U}(E) \times \{0\} \times E)^{\mathrm{U}(E) \times \mathrm{Z}(E)}$$
$$\xrightarrow{r_{\mathfrak{s}^{\mathbf{f}} \times E}} C^{-\xi}(\mathfrak{s}^{\mathbf{f}} \times E; \{0\} \times E)^{\mathrm{Z}(E)}$$

the linear isomorphism specified by the rule

$$f = 1 \otimes r_{\mathfrak{s}^{\mathbf{f}} \times E} f.$$

Write $\mathcal{V}_{\mathfrak{s}^{\mathbf{f}} \times E, \mathbf{e}}$ for the space

$$C^{-\xi}(\mathfrak{s}^{\mathbf{f}} \times E; \{0\} \times (E(\mathbf{e}) \cap \Gamma_E), \{0\} \times (E(\mathbf{e}) \cap \Gamma_E))^{\mathrm{Z}(E)}.$$

In previous notations (see (1) and (19)), we have

$$\mathcal{V}_{\mathfrak{s}^{\mathbf{f}} \times E, \mathbf{e}} = C^{-\xi}(\mathfrak{s}^{\mathbf{f}}; \{0\}) \otimes \mathcal{V}_{E, \mathbf{e}}. \tag{32}$$

Lemma 4.11. *The composition map $r_{\mathfrak{s}^{\mathbf{f}} \times E} \circ \theta^*$ sends $\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}}$ into $\mathcal{V}_{\mathfrak{s}^{\mathbf{f}} \times E, \mathbf{e}}$, and the following diagram*

$$
\begin{array}{ccc}
\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}} & \xrightarrow{r_{\mathfrak{s}^{\mathbf{f}} \times E} \circ \theta^*} & \mathcal{V}_{\mathfrak{s}^{\mathbf{f}} \times E, \mathbf{e}} \\
{\scriptstyle \epsilon_{\mathfrak{s}}} \downarrow & & \downarrow {\scriptstyle \epsilon_{\mathfrak{s}^{\mathbf{f}}, 1 - \mathrm{ad}(\mathbf{h}/2)} - \epsilon_{E, \mathbf{h}/2}} \\
\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}} & \xrightarrow{r_{\mathfrak{s}^{\mathbf{f}} \times E} \circ \theta^*} & \mathcal{V}_{\mathfrak{s}^{\mathbf{f}} \times E, \mathbf{e}}
\end{array}
$$

commutes.

Proof. The first assertion follows by noting that both θ^* and $r_{\mathfrak{s}^{\mathbf{f}}, E}$ commute with the partial Fourier transform along E. The second assertion follows from Lemma 4.10. $\qquad\square$

Lemma 4.12. *Assume that $\dim_A(E) \geq 2$. Then the vector field $\epsilon_{\mathfrak{s}^{\mathbf{f}}, 1 - \mathrm{ad}(\mathbf{h}/2)}$ acts semisimply on $C^{-\xi}(\mathfrak{s}^{\mathbf{f}}; \{0\})$, and all its eigenvalues are real numbers $< -\frac{1}{2} \dim_{\mathbb{R}} \mathfrak{s}$.*

Proof. The condition $\dim_A(E) \geq 2$ implies that $\mathfrak{s} \neq \{0\}$.

We view $\mathfrak{s}$ as a $\mathfrak{sl}_2(\mathbb{R})$-module via the adjoint representation and the standard triple $\{\mathbf{h}, \mathbf{e}, \mathbf{f}\}$. We shall prove that the analog of Lemma 4.12 holds for any finite dimensional nonzero $\mathfrak{sl}_2(\mathbb{R})$-module F. Without loss of generality, we may assume that F is irreducible of real dimension $r + 1$. Then

$$
\epsilon_{F^{\mathbf{f}}, 1 - \mathbf{h}/2} = (1 + r/2) \epsilon_{F^{\mathbf{f}}},
$$

which clearly acts semisimply on $C^{-\xi}(F^{\mathbf{f}}; \{0\})$, with all its eigenvalues real numbers $\leq -(1 + r/2) = -\frac{1}{2} \dim_{\mathbb{R}} F - \frac{1}{2} < -\frac{1}{2} \dim_{\mathbb{R}} F$. $\qquad\square$

In view of (32), Lemma 4.8 will follow from Lemma 4.11, Lemma 4.12, together with Propositions 3.2 and 3.3.

5. Reduction to the null cone

We first recall the following elementary (and well-known) lemma.

Lemma 5.1. *Let H be a Lie group acting smoothly on a manifold M. Let χ_H be a continuous character on H. If $C_{\chi_H}^{-\infty}(M) = 0$, then $C_{\chi_H}^{-\infty}(M') = 0$ for any open submanifold M' of the form $\phi^{-1}(N')$, where $\phi: M \to N$ is an H-equivariant smooth map, N is a manifold with trivial H-action, and N' is an open submanifold of N.*

Recall that (A, τ) is a commutative involutive algebra, and E is a hermitian A-module, as well as other notations from Section 3. The following result may be considered as a case of Harish-Chandra descent.

Proposition 5.2. *Assume that A is simple, $\dim_A(E) \geq 1$, and for all commutative involutive algebra A' and all hermitian A'-module E',*

(33) $$\dim_{A'}(E') < \dim_A(E) \quad implies \quad C_{\chi_{E'}}^{-\infty}(\mathrm{U}(E') \times E') = 0.$$

Then every $f \in C_{\chi_E}^{-\infty}(\mathrm{U}(E) \times E)$ is supported in $(\mathrm{Z}(E)\mathcal{U}_E) \times E$, where $\mathcal{U}_E$ is the set of unipotent elements in $\mathrm{U}(E)$.

Proof. Extend the involution τ on A to $\mathfrak{gl}_A(E)$ (still denoted by τ), by requiring that

$$\langle xu, v \rangle_E = \langle u, x^\tau v \rangle_E, \quad x \in \mathfrak{gl}_A(E), \, u, v \in E.$$

Now let x be a semisimple element in $\mathrm{U}(E) \setminus \mathrm{Z}(E)$. Let A' be the $\mathbb{R}$-subalgebra of $\mathfrak{gl}_A(E)$ generated by A, x and x^τ, which is a commutative involutive algebra. Put $E' = E$, but viewed as an A'-module. Define a map

$$\langle \, , \, \rangle_{E'} : E' \times E' \to A'$$

by requiring that

$$\mathrm{tr}_{A'/\mathbb{R}}(a \langle u, v \rangle_{E'}) = \mathrm{tr}_{A/\mathbb{R}}(\langle au, v \rangle_E), \quad a \in A', \, u, v \in E.$$

Then E' becomes a hermitian A'-module, with

$$\dim_{A'}(E') < \dim_A(E),$$

and $\tilde{\mathrm{U}}(E')$ coincides with the subgroup of $\tilde{\mathrm{U}}(E)$ consisting of all (g, δ) such that

$$gxg^{-1} = \begin{cases} x, & \text{if } \delta = 1, \\ x^\tau, & \text{if } \delta = -1. \end{cases}$$

For any $y \in \mathrm{U}(E')$, denote by $J(y)$ the determinant of the $\mathbb{R}$-linear map

$$1 - \mathrm{Ad}_{y^{-1}} : \mathfrak{u}(E)/\mathfrak{u}(E') \to \mathfrak{u}(E)/\mathfrak{u}(E').$$

Note that Ad_y preserves a non-degenerate real quadratic form on $\mathfrak{u}(E)/\mathfrak{u}(E')$. This implies that J is a $\tilde{\mathrm{U}}(E')$-invariant function (under the action (25)). Put

$$\mathrm{U}(E')^\circ := \{y \in \mathrm{U}(E') \mid J(y) \neq 0\},$$

which contains $x\mathcal{U}_{E'}$. The map

$$\pi : \tilde{\mathrm{U}}(E) \times (\mathrm{U}(E')^\circ \times E') \quad \to \quad \mathrm{U}(E) \times E,$$
$$(\tilde{g}, y, v) \quad \mapsto \quad \tilde{g}(y, v)$$

is a submersion. Therefore we have a well defined restriction map ([JSZ, Lemma 4.4])

$$r_{E,E'} : C_{\chi_E}^{-\infty}(\mathrm{U}(E) \times E) \to C_{\chi_{E'}}^{-\infty}(\mathrm{U}(E')^\circ \times E'),$$

which is specified by the rule

$$\pi^*(f) = \chi_E \otimes r_{E,E'}(f).$$

The assumption (33) and Lemma 5.1 imply that the later space is zero. Thus every $f \in C^{-\infty}_{\chi_E}(\mathrm{U}(E) \times E)$ vanishes on the image of π. As x is arbitrary, the proposition follows. $\square$

The remaining part of this section is devoted to the proof of the following

Proposition 5.3. *Assume that A is simple, $\dim_A(E) \geq 1$, and for all commutative involutive algebra A' and all hermitian A'-module E',*

(34) $$\dim_{A'}(E') < \dim_A(E) \quad \text{implies} \quad C^{-\xi}_{\chi_{E'}}(\mathfrak{u}(E') \times E') = 0.$$

Then every $f \in C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_E) \times \Gamma_E$.

Similar to the proof of Proposition 5.2, we show that every $f \in C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_E) \times E$. We are left to show that f is also supported in $\mathfrak{u}(E) \times \Gamma_E$.

Fix
$$t \in (A^{\times})^{\{1,\tau\}} := \{a \in A^{\times} \mid a^{\tau} = a\},$$

and set
$$E(t) := \{v \in E \mid \langle v, v \rangle_E = t\}.$$

Fix $v_0 \in E(t)$ (when it is nonempty), and put
$$E' := \{v \in E \mid \langle v, v_0 \rangle_E = 0\}.$$

Then
$$E = E' \oplus Av_0$$

is an orthogonal decomposition of hermitian A-modules. We identify $\tilde{\mathrm{U}}(E')$ with a subgroup of $\tilde{\mathrm{U}}(E)$ via the embedding

(35) $$(g, \delta) \mapsto \left(\begin{bmatrix} \delta g & 0 \\ 0 & \tau_\delta \end{bmatrix}, \delta \right),$$

where $\tau_\delta : Av_0 \to Av_0$ is the $\mathbb{R}$-linear map given by
$$\tau_\delta(av_0) = \begin{cases} av_0, & \text{if } \delta = 1, \\ -a^{\tau} v_0, & \text{if } \delta = -1. \end{cases}$$

Then $\tilde{\mathrm{U}}(E')$ is precisely the stabilizer of v_0 in $\tilde{\mathrm{U}}(E)$.

Let $\tilde{\mathrm{U}}(E')$ act on $\mathfrak{u}(E)$ by way of the action of $\tilde{\mathrm{U}}(E)$.

Lemma 5.4. *With notations as above, we have*
$$\mathfrak{u}(E) \cong \mathfrak{u}(E') \times E' \times \mathfrak{u}(Av_0)$$

as $\mathbb{R}$-linear representations of $\tilde{\mathrm{U}}(E')$. Here $\tilde{\mathrm{U}}(E')$ acts on $\mathfrak{u}(Av_0)$ trivially.

Proof. Denote by $p_{E'}$ the projection to the first factor according to the orthogonal decomposition

$$E = E' \oplus Av_0.$$

For any

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in \mathfrak{u}(E),$$

set

$$\pi_1(x) := x_{11}, \quad \pi_2(x) := p_{E'}(xv_0), \quad \text{and} \quad \pi_3(x) := x_{22}.$$

It is routine to check that the map

$$\pi_1 \times \pi_2 \times \pi_3 : \mathfrak{u}(E) \to \mathfrak{u}(E') \times E' \times \mathfrak{u}(Av_0)$$

is injective and $\tilde{U}(E')$-intertwining. By comparing the dimension, we see that the map is an isomorphism.

$\square$

We now finish the proof of Proposition 5.3. Define a map

$$\begin{aligned} \rho_t : (A^\times)^{\{1,\tau\}} \times \mathfrak{u}(E) \times E(t) &\to \mathfrak{u}(E) \times E, \\ (a, x, v) &\mapsto (x, av). \end{aligned}$$

Let $\tilde{U}(E)$ act on $(A^\times)^{\{1,\tau\}}$ trivially. Note that $\tilde{U}(E)$ preserves $E(t)$, and is transitive on $E(t)$ by Witt's lemma. The map ρ_t is a $\tilde{U}(E)$-equivariant submersion, and hence defines a pull back map

$$\rho_t^* : C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E) \to C_{\chi_E}^{-\xi}((A^\times)^{\{1,\tau\}} \times \mathfrak{u}(E) \times E(t)).$$

We have

$$\begin{aligned} &C_{\chi_E}^{-\xi}((A^\times)^{\{1,\tau\}} \times \mathfrak{u}(E) \times E(t)) = 0 \\ \iff &C_{\chi_E}^{-\xi}((\mathfrak{u}(E) \times E(t)) = 0 \qquad \text{the localization principle} \\ \iff &C_{\chi_{E'}}^{-\xi}(\mathfrak{u}(E)) = 0 \qquad \text{Frobenius reciprocity} \\ \iff &C_{\chi_{E'}}^{-\xi}(\mathfrak{u}(E') \times E' \times \mathfrak{u}(Av_0)) = 0 \qquad \text{Lemma 5.4} \\ \iff &C_{\chi_{E'}}^{-\xi}(\mathfrak{u}(E') \times E') = 0. \qquad \text{the localization principle} \end{aligned}$$

For a proof of Frobenius reciprocity, see [AG2, Theorem C.3.3] for example. As the last equality holds by assumption (34), the first equality holds. Consequently, every $f \in C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E)$ vanishes on the image of ρ_t. As $t \in (A^\times)^{\{1,\tau\}}$ is arbitrary, and we conclude that f is supported in $\mathfrak{u}(E) \times \Gamma_E$.

6. Proof of Theorem B

Let (A, τ) be a commutative involutive algebra, and let E be a hermitian A-module, as in Section 3. Recall the group $\tilde{U}(E)$ and the quadratic character χ_E of $\tilde{U}(E)$, as in (23) and (24).

Lemma 6.1. *Write*

$$A = A_1 \times A_2 \times \cdots \times A_r$$

as a product of simple commutative involutive algebras. Set

$$E_j = A_j \otimes_A E,$$

which is canonically an hermitian A_j-module.

(a) *If*

(36) $$C_{\chi_{E_i}}^{-\infty}(U(E_i) \times E_i) = 0 \quad \text{for all } i = 1, 2, \cdots, r,$$

   then

$$C_{\chi_E}^{-\infty}(U(E) \times E) = 0.$$

(b) *If*

$$C_{\chi_{E_i}}^{-\xi}(\mathfrak{u}(E_i) \times E_i) = 0 \quad \text{for all } i = 1, 2, \cdots, r,$$

   then

$$C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E) = 0.$$

Proof. Let us prove Part (a). Part (b) is proved similarly. Note that

$$E = E_1 \times E_2 \times \cdots \times E_r,$$

and

$$U(E) = U(E_1) \times U(E_2) \times \cdots \times U(E_r).$$

Recall the following elementary fact (c.f. [AGS1, Proposition 3.1.5]). Let H_i be a Lie group acting smoothly on a manifold M_i, and let H_i' be a subgroup of H_i, $i = 1, 2, \cdots, r$. If

$$C^{-\infty}(M_i)^{H_i'} = C^{-\infty}(M_i)^{H_i} \quad \text{for all } i = 1, 2, \cdots, r,$$

then

(37) $$C^{-\infty}(M_1 \times M_2 \times \cdots \times M_k)^{H_1' \times H_2' \times \cdots \times H_r'}$$
$$= C^{-\infty}(M_1 \times M_2 \times \cdots \times M_k)^{H_1 \times H_2 \times \cdots \times H_r}.$$

Note that (36) is equivalent to

$$C^{-\infty}(U(E_i) \times E_i)^{U(E_i)} = C^{-\infty}(U(E_i) \times E_i)^{\tilde{U}(E_i)}.$$

By (37), we have

$$C^{-\infty}(\mathrm{U}(E) \times E)^{\mathrm{U}(E_1) \times \mathrm{U}(E_2) \times \cdots \times \mathrm{U}(E_r)}$$
$$= C^{-\infty}(\mathrm{U}(E) \times E)^{\tilde{\mathrm{U}}(E_1) \times \tilde{\mathrm{U}}(E_2) \times \cdots \times \tilde{\mathrm{U}}(E_r)}.$$

Now Part (a) of the lemma follows by noting that, as operators on $\mathrm{U}(E) \times E$, the group $\tilde{\mathrm{U}}(E)$ coincides with the subgroup of $\tilde{\mathrm{U}}(E_1) \times \tilde{\mathrm{U}}(E_2) \times \cdots \times \tilde{\mathrm{U}}(E_r)$ consisting of elements of the form

$$((g_1, \delta), (g_2, \delta), \cdots, (g_r, \delta)).$$

$\square$

Lemma 6.2. *Let $\mathbf{H}$ be a reductive linear algebraic group defined over $\mathbb{R}$, with an algebraic action on a finite dimensional real vector space F. Let $\chi_{\mathbf{H}}$ be a (continuous) quadratic character of $\mathbf{H}(\mathbb{R})$. Then*

$$C_{\chi_{\mathbf{H}}}^{-\infty}(F) = 0 \quad \text{if and only if} \quad C_{\chi_{\mathbf{H}}}^{-\xi}(F) = 0.$$

See [AG2, Theorem 4.0.8] for a proof, which uses geometry invariant theory.

Proposition 6.3. *One has that*

$$C_{\chi_E}^{-\infty}(\mathfrak{u}(E) \times E) = 0.$$

Proof. By Lemma 6.2, we only need to prove that

$$(38) \qquad\qquad C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E) = 0.$$

We prove (38) by induction on $\dim_A(E)$. When $\dim_A(E) = 0$, we have $\tilde{\mathrm{U}}(E) = \{\pm 1\}$ and so (38) is trivially true. So assume that $\dim_A(E) \geq 1$, and that (38) holds for all commutative involutive algebra A' and all hermitian A'-module E' with $\dim_{A'}(E') < \dim_A(E)$.

By Lemma 6.1, we may further assume that A is simple. By Proposition 5.3, we see that every $f \in C_{\chi_E}^{-\xi}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}) \times \Gamma_E$. Proposition 4.1 then implies that $f = 0$. $\square$

Proposition 6.4. *One has that*

$$C_{\chi_E}^{-\infty}(\mathrm{U}(E) \times E) = 0.$$

Proof. Again, we prove by induction on $\dim_A(E)$. When $\dim_A(E) = 0$, the proposition is trivially true. So assume that $\dim_A(E) \geq 1$, and that the proposition holds for all commutative involutive algebra A' and all hermitian A'-module E' with $\dim_{A'}(E') < \dim_A(E)$.

By Lemma 6.1, we may further assume that A is simple. By Proposition 5.2, we see that every $f \in C_{\chi_E}^{-\infty}(\mathrm{U}(E) \times E)$ is supported in $(\mathrm{Z}(E)\mathcal{U}_E) \times E$.

Define a $\tilde{\mathrm{U}}(E)$-equivariant map

$$\begin{aligned} \rho_E : \mathrm{Z}(E) \times \mathfrak{su}(E) \times E &\to \mathrm{U}(E) \times E, \\ (z, x, v) &\mapsto (z \exp(x), v). \end{aligned}$$

As is well-known, ρ_E is submersive from $\mathrm{Z}(E) \times \mathcal{N}_E \times E$ onto $(\mathrm{Z}(E)\mathcal{U}_E) \times E$. Therefore it yields an injective pull back map

$$C_{\chi_E}^{-\infty}(\mathrm{U}(E) \times E; (\mathrm{Z}(E)\mathcal{U}_E) \times E)$$

$$\xrightarrow{\rho_E^*} C_{\chi_E}^{-\infty}(\mathrm{Z}(E) \times \mathfrak{su}(E) \times E; \mathrm{Z}(E) \times \mathcal{N}_E \times E).$$

To finish the proof, it suffices to show that

$$C_{\chi_E}^{-\infty}(\mathrm{Z}(E) \times \mathfrak{su}(E) \times E) = 0.$$

Since $\tilde{\mathrm{U}}(E)$ acts on $\mathrm{Z}(E)$ trivially, by the localization principle, this is equivalent to

$$C_{\chi_E}^{-\infty}(\mathfrak{su}(E) \times E) = 0.$$

Again, since $\tilde{\mathrm{U}}(E)$ acts on $\mathfrak{z}(E)$ trivially and $\mathfrak{u}(E) = \mathfrak{z}(E) \oplus \mathfrak{su}(E)$, this is equivalent to

$$C_{\chi_E}^{-\infty}(\mathfrak{u}(E) \times E) = 0,$$

which is asserted by Proposition 6.3. $\qquad\square$

Theorem 6.5. *Let E be a hermitian A-module and assume that A is simple. Let $v_0 \in E \setminus \Gamma_E$, and define*

$$E' := \{v \in E \mid \langle v, v_0 \rangle_E = 0\},$$

which is a hermitian A-module. Identify $\tilde{\mathrm{U}}(E')$ with the stabilizer of v_0 in $\tilde{\mathrm{U}}(E)$ via the embedding (35), and let $\tilde{\mathrm{U}}(E')$ act on $\mathrm{U}(E)$ by way of the action of $\tilde{\mathrm{U}}(E)$. Then

$$C_{\chi_{E'}}^{-\infty}(\mathrm{U}(E)) = 0.$$

Proof. As in Section 5, put

$$t := \langle v_0, v_0 \rangle \in (A^\times)^{\{1,\tau\}},$$

and

$$E(t) := \{v \in E \mid \langle v, v \rangle_E = t\},$$

which is an $\tilde{\mathrm{U}}(E)$-homogeneous space.

Fix a $\tilde{\mathrm{U}}(E)$-invariant positive measure $\mu_{E(t)}$ on $E(t)$, and a Lebesgue measure μ_E on E. Define a map

$$J_t : C^{-\infty}(E(t)) \to C^{-\infty}(E)$$

by requiring that the diagram

$$
\begin{array}{ccc}
C^{-\infty}(E(t)) & \xrightarrow{\;\;J_t\;\;} & C^{-\infty}(E) \\
\downarrow{\scriptstyle\cdot\mu_{E(t)}} & & \downarrow{\scriptstyle\cdot\mu_E} \\
D^{-\infty}(E(t)) & \longrightarrow & D^{-\infty}(E)
\end{array}
$$

commutes, where $D^{-\infty}$ stands for the space of distributions, the lower horizontal arrow is the push forward of distributions via the closed embedding $E(t) \hookrightarrow E$, and the vertical arrows are linear isomorphisms given by multiplications of the indicated measures.

Then we have an injective continuous linear map

$$1 \otimes J_t : C_{\chi_E}^{-\infty}(\mathrm{U}(E) \times E(t)) \hookrightarrow C_{\chi_E}^{-\infty}(\mathrm{U}(E) \times E).$$

The later space vanishes by Proposition 6.4, and therefore so does the former one. We finish the proof by using Frobenius reciprocity.

$\square$

We now show that Theorem B is implied by Theorem 6.5. Let A be one of the five simple commutative involutive algebras as in (10), and E be the hermitian A-module as in (12), with $n, q \geq 1$. Let v_0 be the vector in $E \setminus \Gamma_E$ given by

$$
v_0 := \begin{cases} [0, 0, \cdots, 0, 1]^t, & \text{if } A \text{ is of type I,} \\ [0, 0, \cdots, 0, 1, 0, 0, \cdots, 0, 1]^t, & \text{if } A \text{ is of type II.} \end{cases}
$$

Then G' of Theorem B coincides with $\mathrm{U}(E')$ of Theorem 6.5. Define $\sigma_0 := (-x_0, -1)$, where $x_0 \in \mathrm{GL}_\mathbb{R}(E)$ is given by

$$
x_0 := \begin{cases} 1, & \text{if } A = (\mathbb{K}, 1), \\ \bar{}, & \text{if } A = (\mathbb{C}, \bar{}), \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } A = (\mathbb{K} \times \mathbb{K}, \tau_\mathbb{K}). \end{cases}
$$

Then σ_0 is an element of $\tilde{\mathrm{U}}(E) \setminus \mathrm{U}(E)$ fixing v_0, and so is in $\tilde{\mathrm{U}}(E') \setminus \mathrm{U}(E')$. See (23) for the description of $\tilde{\mathrm{U}}(E)$ and (35) for the explicit embedding of $\tilde{\mathrm{U}}(E')$ in $\tilde{\mathrm{U}}(E)$. Theorem B follows from Theorem 6.5 by observing that σ_0 yields the anti-involution σ of $G = \mathrm{U}(E)$, as desired.

7. Theorem B implies Theorem A

This section is devoted to a proof of the following proposition, which says that Theorem B implies Theorem A, in a general setting. The argument is standard.

Proposition 7.1. *Let G be a real reductive group, with a reductive closed subgroup G'. Let σ be a continuous anti-automorphism of G which leaves G' stable. Assume that for every generalized function f on G, the condition*

$$f(gxg^{-1}) = f(x) \quad \text{for all } g \in G'$$

implies

$$f(x^{\sigma}) = f(x).$$

Then for all irreducible Harish-Chandra smooth representation V and V' of G and G', respectively, we have

$$\dim \operatorname{Hom}_{G'}(V, V') \leq 1.$$

We will use the following form of Gelfand-Kazhdan criteria. A more general version is proved in [SZ, Theorem 2.2].

Lemma 7.2. *Let S be a closed subgroup of a real reductive group H, and let σ be a continuous anti-automorphism of H. Assume that every bi S-invariant generalized function on H is σ-invariant. Then for any two irreducible Harish-Chandra smooth representations U_H and V_H of H which are contragredient to each other, one has that*

$$\dim \operatorname{Hom}_S(U_H, \mathbb{C}) \ \dim \operatorname{Hom}_S(V_H, \mathbb{C}) \leq 1.$$

We return to the proof of Proposition 7.1. Set

$$H := G \times G',$$

which contains G as a subgroup. Denote by $S \subset H$ the group G' diagonally embedded in H. For any $x = (g, g') \in H$, set

$$x^{\sigma} := (g^{\sigma}, g'^{\sigma}).$$

Lemma 7.3. *Under the assumption of Proposition 7.1, if f is a bi S-invariant generalized function on H, then*

$$f(x^{\sigma}) = f(x).$$

Proof. The multiplication map

$$
\begin{aligned}
m_H : S \times G \times S &\to H \\
(s_1, g, s_2) &\mapsto s_1 g s_2
\end{aligned}
$$

is a surjective submersion. Let f be a bi S-invariant generalized function on H. Then its pull back has the form

$$m_H^*(f) = 1 \otimes f_G \otimes 1, \quad \text{with } f_G \in C^{-\infty}(G).$$

By considering the commutative diagram

$$
\begin{array}{ccc}
S \times G \times S & \xrightarrow{\ m_H\ } & H \\
{\scriptstyle \mathrm{Ad}_s \times \mathrm{Ad}_s \times \mathrm{Ad}_s}\Big\downarrow & & {\scriptstyle \mathrm{Ad}_s}\Big\downarrow \\
S \times G \times S & \xrightarrow{\ m_H\ } & H\,,
\end{array}
$$

for all $s \in S$, we conclude that f_G is invariant under the adjoint action of G'. By the assumption of Proposition 7.1, we conclude that f_G is σ-invariant.

Set

$$(s_1, g, s_2)^\sigma := (s_2^\sigma, g^\sigma, s_1^\sigma), \quad (s_1, g, s_2) \in S \times G \times S.$$

Then $1 \otimes f_G \otimes 1 \in C^{-\infty}(S \times G \times S)$ is also σ-invariant. We conclude that f is σ-invariant by appealing to the commutative diagram

$$
\begin{array}{ccc}
S \times G \times S & \xrightarrow{\ m_H\ } & H \\
{\scriptstyle \sigma}\Big\downarrow & & {\scriptstyle \sigma}\Big\downarrow \\
S \times G \times S & \xrightarrow{\ m_H\ } & H\,.
\end{array}
$$

$\hfill\square$

Lemma 7.4. *We are under the assumption of Proposition 7.1. Let (V_H, ρ) be an irreducible Harish-Chandra smooth representation of H.*

(a) *Set*

$$\rho_{-\sigma}(h) := \rho(h^{-\sigma}).$$

Then $(V_H, \rho_{-\sigma})$ is an irreducible Harish-Chandra smooth representation of H which is contragredient to (V_H, ρ).

(b) *We have*

$$\dim \mathrm{Hom}_S(V_H, \mathbb{C}) \le 1.$$

Proof. Denote by

$$\chi_\rho \in C^{-\infty}(H)$$

the character of (V_H, ρ). Then its contragredient representation has character $\chi_\rho(h^{-1})$.

It is clear that $(V_H, \rho_{-\sigma})$ is an irreducible Harish-Chandra smooth representation, with character $\chi_\rho(h^{-\sigma})$. Note that the assumption of Proposition 7.1 easily implies that every generalized function on H is σ-invariant provided it is invariant under the adjoint action of H. Since a character is always invariant under the adjoint action, we conclude that

$$\chi_\rho(h^{-1}) = \chi_\rho(h^{-\sigma}).$$

Part (a) then follows from the well-known fact that an irreducible Harish-Chandra smooth representation is determined by its character.

For Part (b), denote by U_H the irreducible Harish-Chandra smooth representation which is contragredient to V_H. Lemma 7.2 and Lemma 7.3 imply that

$$\dim \operatorname{Hom}_S(U_H, \mathbb{C}) \; \dim \operatorname{Hom}_S(V_H, \mathbb{C}) \leq 1.$$

Now Part (a) clearly implies that

$$\dim \operatorname{Hom}_S(U_H, \mathbb{C}) = \dim \operatorname{Hom}_S(V_H, \mathbb{C}).$$

We therefore conclude that $\dim \operatorname{Hom}_S(V_H, \mathbb{C}) \leq 1$. $\hfill\square$

We now finish the proof of Proposition 7.1. Denote by U' the irreducible Harish-Chandra smooth representation of G' which is contragredient to V'. Set

$$V_H := V \widehat{\otimes} U',$$

which is an irreducible Harish-Chandra smooth representation of H. As usual, we have an obvious linear embedding

$$\operatorname{Hom}_{G'}(V, V') \hookrightarrow \operatorname{Hom}_S(V_H, \mathbb{C}).$$

The later space is at most one dimensional by Lemma 7.4, and so is the former.

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